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A POLYNOMIAL REALIZATION OF THE HOPF ALGEBRA OF UNIFORM BLOCK PERMUTATIONS.

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Abstract. We provide a polynomial realization of the Hopf algebra **UBP** of uniform block permutations defined by Aguiar and Orellana [J. Alg. Combin. 28 (2008), 115-138]. We describe an embedding of the dual of the Hopf algebra **WQSym** into **UBP**, and as a consequence, obtain a polynomial realization of it.

Contents

1. Introduction

For several years, Hopf algebras based on combinatorial objects have been thoroughly investigated. Examples include the Malvenuto-Reutenauer Hopf algebra **FQSym** whose bases are indexed by permutations [1,3,8], the Loday-Ronco Hopf algebra **PBT** whose bases are indexed by planar binary trees, [6], the Hopf algebra of free symmetric functions **FSym** whose bases are indexed by standard Young tableaux [3, 12].

The product and the coproduct, which define the structure of a combinatorial Hopf algebra, are, in general, rules of composition and decomposition given by combinatorial algorithms such as the shuffle and the deconcatenation of permutations, as in **FQSym** [3], of packed words, as in **WQSym** [10] or of parking functions, as in **PQSym** [11]. These can also be given by the disjoint union and the admissible cuts of rooted trees, as in the Connes-Kreimer Hopf algebra, [2], or by concatenation and subgraphs/contractions of graphs [9].

Computing with these structures can be difficult, and polynomial realizations can bring up important simplifications.

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The idea is to encode the combinatorial objects by polynomials, in such a way that the product of the combinatorial Hopf algebra become the ordinary product of polynomials, and that the coproduct be given by the disjoint union of alphabets, endowed with some extra structure such as an order relation. This can be done, *e.g.,* for **FQSym** [3], **WQSym** [10], **PQSym** [11], Connes-Kreimer [5] and the one of diagrams [4].

In this paper, we realize the Hopf algebra **UBP** of uniform block permutations introduced by Aguiar and Orellana [7]. We begin by recalling the preliminary notions on uniform block permutations in Section 2. We recall, in Section 3, the Hopf algebra structure of uniform block permutations, we translate this structure via some decomposition of uniform block permutations into pairs consisting of a permutation and a set partition, and we describe the dual structure. Using polynomial realizations of Hopf algebras based on permutations and set partitions, we realize this algebra in terms of noncommutative polynomials in infinitely many bi-letters in Section 4. Finally, we obtain the Hopf algebra **WQSym** as a quotient of **UBP** and its dual as a subalgebra of $\mathbf{UBP}^*$.

2. Notations and background

We denote by $[n]$ the set $\{1, 2, \cdots, n\}$. All algebras will be on some field $\mathbb{K}$ of characteristic zero.

2.1. Words.

Let $A := \{a_1 < a_2 < \cdots\}$ be a totally ordered infinite alphabet and A^* the free monoid generated by A. For a word $w \in A^*$, we denote by $|w|$ the length of w. The empty word (of length 0) is denoted by ε. The alphabet $\mathrm{Alph}(w)$ of w is the set of letters occuring in w.

We say that (i, j) is an *inversion* of w if $i < j$ and $w_i > w_j$.

For $u, v \in A^*$, the *shuffle product* $⧢$ is recursively defined by

$$u ⧢ v := \begin{cases} u & \text{if } v = \varepsilon \\ v & \text{if } u = \varepsilon \\ u_1(u' ⧢ v) + v_1(u ⧢ v') & \text{otherwise} \end{cases} \tag{2.1}$$

where $u = u_1u'$ and $v = v_1v'$ with $u_1, v_1 \in A$ and $u', v' \in A^*$.

We shall need the following expression of the shuffle:

Proposition 2.1. *For all $u, v \in A^*$ and for all $0 \le k \le |u| + |v|$:*

$$u ⧢ v = \sum_{\substack{i+j=k \\ 0\le i\le |u| \\ 0\le j\le |v|}} (u_1 \cdots u_i ⧢ v_1 \cdots v_j) \cdot (u_{i+1} \cdots u_{|u|} ⧢ v_{j+1} \cdots v_{|v|}) \tag{2.2}$$

Proof. We may assume that $\mathrm{Alph}(u)$ and $\mathrm{Alph}(v)$ are disjoint. The proof proceeds by induction on the length of the prefix in $u ⧢ v$.

Let k be an integer such that $0 \le k \le |u| + |v|$. Then the set of all prefixes of length k of $u ⧢ v$ is equal to the set of words of the product $u' ⧢ v'$ with u' (*resp.* v') is a prefix of u (*resp.* v) and $|u'| + |v'| = k$. □

More conceptually, if $\delta(w) = \sum_{uv=w} u \otimes v$ is the deconcatenation coproduct, and $\mu :\ u \otimes v \mapsto uv$ the concatenation product, then δ is a morphism for $⧢$. Define a projection by $\pi_{k,l}(u \otimes v) = u \otimes v$ if $|u| = k$ and $|v| = l$, and $\pi_{k,l}(u \otimes v) = 0$ otherwise. Obviously, $\pi_{k,l} \circ \delta(w) = w$ if $k + l = |w|$. The proposition is equivalent to $\mu \circ \pi_{k,n-k} \circ \delta(u ⧢ v) = \mu \circ \pi_{k,n-k}(\delta(u) ⧢ \delta(v)) = u ⧢ v$.

Example 2.2.

$$\begin{aligned}a_1a_2a_3 \shuffle a_1a_3 =& a_1a_2a_3a_1a_3 + 2a_1a_2a_1a_3a_3 + 4a_1a_1a_2a_3a_3 + 2a_1a_1a_3a_2a_3 + a_1a_3a_1a_2a_3\\ =& a_1(a_2a_3 \shuffle a_1a_3) + a_1(a_1a_2a_3 \shuffle a_3)\\ =& a_1a_2(a_3 \shuffle a_1a_3) + 2a_1a_1(a_2a_3 \shuffle a_3) + a_1a_3a_1a_2a_3\\ =& a_1a_2a_3a_1a_3 + 2(a_1a_2 \shuffle a_1)a_3a_3 + (a_1 \shuffle a_1a_3)a_2a_3\\ =& (a_1a_2a_3 \shuffle a_1)a_3 + (a_1a_2 \shuffle a_1a_3)a_3.\end{aligned}$$

2.2. **Permutations.** We denote by $\mathfrak{S}_n$ the set of permutations of size n and we set $\mathfrak{S} = \cup_{n\geq 0}\mathfrak{S}_n$. The *shifted shuffle product* $\overline{\shuffle}$ is defined by

$$\sigma_1 \,\overline{\shuffle}\, \sigma_2 = \sigma_1 \shuffle \sigma_2[|\sigma_1|] \tag{2.3}$$

where $\sigma[k]$ is the word obtained by adding k to each letter of σ.

Example 2.3.

$$\begin{aligned}12 \,\overline{\shuffle}\, 21 =& 12 \shuffle 43\\ =& 1243 + 1423 + 4123 + 1432 + 4132 + 4312.\end{aligned}$$

The *standardization* is a process that associates a permutation with a word. The standardized of w is defined as the permutation having the same inversions as w.

Example 2.4.

$$\mathrm{std}(a_2a_3a_2a_1a_6a_1a_4) = 3541726.$$

The *convolution product* $*$ is defined by

$$\sigma_1 * \sigma_2 = \sum_{\substack{w=u\cdot v\in\mathfrak{S}\\ \mathrm{std}(u)=\sigma_1\\ \mathrm{std}(v)=\sigma_2}} w. \tag{2.4}$$

Example 2.5.

$$12 * 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421.$$

2.3. **The Hopf algebra of set partitions.** We denote by $\mathcal{P}_n$ the set of set partitions of $[n]$ and we set $\mathcal{P} = \cup_{n\geq 0}\mathcal{P}_n$. Given a collection of disjoint sets of integers, we denote by $\mathrm{std}(\mathcal{E})$ the standardized of $\mathcal{E}$ obtained by numbering each integer a in $\mathcal{E}$ by one plus the number of integers in $\mathcal{E}$ smaller than a.

Example 2.6.

$$\mathrm{std}\left(\big\{\{1,6\},\{9,13\},\{3,5,12\}\big\}\right) = \big\{\{1,4\},\{5,7\},\{2,3,6\}\big\}.$$

The $\mathbb{K}$-vector space spanned by the set partitions $\mathcal{P}$ can be endowed with a Hopf algebra structure. The product, denoted by $\times$, is obtained by the shifted union of two set partitions and the coproduct is defined for a set partition $\mathcal{A} = \{A_1, \cdots, A_k\}$ by

$$\Delta(\mathcal{A}) = \sum_{H\subset\{1,2,\cdots,k\}} \mathrm{std}\,\Big(\bigcup_{i\in H}\{A_i\}\Big) \otimes \mathrm{std}\,\Big(\bigcup_{\substack{1\leq i\leq k\\ i\notin H}}\{A_i\}\Big). \tag{2.5}$$

Example 2.7.

$$\big\{\{1,3\},\{2\},\{4\}\big\} \times \big\{\{1\},\{2,3\}\big\} = \big\{\{1,3\},\{2\},\{4\},\{5\},\{6,7\}\big\},$$

and, by setting $\mathcal{A} = \big\{\{1,3\},\{2,5\},\{4\}\big\}$,

$$\begin{aligned}\Delta(\mathcal{A}) =& \big\{\{\}\big\} \otimes \mathcal{A} + \big\{\{1,2\}\big\} \otimes \big\{\{1,3\},\{2\}\big\} + \big\{\{1,2\}\big\} \otimes \big\{\{1,2\},\{3\}\big\}\\ &+ \big\{\{1\}\big\} \otimes \big\{\{1,3\},\{2,4\}\big\} + \big\{\{1,3\},\{2,4\}\big\} \otimes \big\{\{1\}\big\}\\ &+ \big\{\{1,2\},\{3\}\big\} \otimes \big\{\{1,2\}\big\} + \big\{\{1,3\},\{2\}\big\} \otimes \big\{\{1,2\}\big\} + \mathcal{A} \otimes \big\{\{\}\big\}.\end{aligned}$$

This algebra admits a polynomial realization [7]. Let B be a noncommutative alphabet and let $\mathcal{A}$ be a set partition. We say that a word $w \in B^*$ is $\mathcal{A}$-compatible if

$$(\forall A \in \mathcal{A})(\forall i, j \in A) \quad w_i = w_j. \tag{2.6}$$

In other words, if i and j are in the same block in $\mathcal{A}$, then $w_i = w_j$. We define

$$P_{\mathcal{A}}(B) = \sum_{\substack{w \in B^* \\ w \text{ is } \mathcal{A}-\text{compatible}}} w. \tag{2.7}$$

Example 2.8.

$$P_{\{\{1\}\}} = \sum_i b_i$$

$$P_{\{\{1,2\}\}} = \sum_i b_i b_i, \quad P_{\{\{1\},\{2\}\}} = \sum_{i,j} b_i b_j$$

$$P_{\{\{1,2,3\}\}} = \sum_i b_i b_i b_i, \quad P_{\{\{1,2\},\{3\}\}} = \sum_{i,j} b_i b_i b_j, \quad P_{\{\{1\},\{2\},\{3\}\}} = \sum_{i,j,k} b_i b_j b_k.$$

Proposition 2.9. *The polynomials $P_{\mathcal{A}}(B)$ provide a realization of the Hopf algebra based on set partitions $\mathcal{P}$. That is to say, given two set partitions $\mathcal{A}$ and $\mathcal{B}$*

$$P_{\mathcal{A}} \cdot P_{\mathcal{B}} = P_{\mathcal{A} \times \mathcal{B}}\,. \tag{2.8}$$

Let B' be an alphabet isomorphic to the alphabet B. If we allow B and B' to commute and identify $T(B)U(B')$ with $T \otimes U$,

$$\Delta(P_{\mathcal{A}}) = P_{\mathcal{A}}(B + B') \tag{2.9}$$

where $B + B'$ denotes the disjoint union of B and B'.

Example 2.10.

$$\begin{aligned} P_{\{\{1,3\},\{2\},\{4\}\}}(B) \cdot P_{\{\{1\},\{2,3\}\}}(B) &= \sum_{i,j,k} b_i b_j b_i b_k \sum_{i,j} b_i b_j b_j \\ &= \sum_{i,j,k,l,m} b_i b_j b_i b_k b_l b_m b_m \\ &= P_{\{\{1,3\},\{2\},\{4\},\{5\},\{6,7\}\}}(B) \end{aligned}$$

We set $A_1 = \{1,3\}$, $A_2 = \{2,5\}$, $A_3 = \{4\}$, and $\mathcal{A} = \big\{A_1, A_2, A_3\big\}$:

$$\begin{aligned} P_{\mathcal{A}}(B + B') = &\sum_{i,j,k} b_i b_j b_i b_k b_j + \sum_{i,j,k} b'_i b_j b'_i b_k b_j + \sum_{i,j,k} b_i b'_j b_i b_k b'_j + \sum_{i,j,k} b_i b_j b_i b'_k b_j \\ &+ \sum_{i,j,k} b'_i b'_j b'_i b_k b'_j + \sum_{i,j,k} b'_i b_j b'_i b'_k b_j + \sum_{i,j,k} b_i b'_j b_i b'_k b'_j + \sum_{i,j,k} b'_i b'_j b'_i b'_k b'_j \\ = &P_{\mathcal{A}}(B) + P_{\{A_2,A_3\}}(B) P_{\{A_1\}}(B') + P_{\{A_1,A_3\}}(B) P_{\{A_2\}}(B') \\ &+ P_{\{A_1,A_2\}}(B) P_{\{A_3\}}(B') + P_{\{A_3\}}(B) P_{\{A_1,A_2\}}(B') \\ &+ P_{\{A_2\}}(B) P_{\{A_1,A_3\}}(B') + P_{\{A_1\}}(B) P_{\{A_2,A_3\}}(B') + P_{\mathcal{A}}(B'). \end{aligned}$$

2.4. The Hopf algebras FQSym and FQSym*.

The Hopf algebra of free quasi-symmetric functions [3] is the $\mathbb{K}$-vector space generated by the family $\{\mathbf{F}_\sigma\}_{\sigma \in \mathfrak{S}}$

$$\mathbf{F}_\sigma := \sum_{w\colon \mathrm{std}(w) = \sigma^{-1}} w. \tag{2.10}$$

The product rule is

$$\mathbf{F}_{\sigma_1} \cdot \mathbf{F}_{\sigma_2} := \sum_{\tau \in \sigma_1 \,\overline{\sqcup\!\sqcup}\, \sigma_2} \mathbf{F}_\tau, \tag{2.11}$$

and the coproduct is defined by

$$\Delta(\mathbf{F}_\sigma) := \mathbf{F}_\sigma(A \oplus B) = \sum_{u \cdot v = \sigma} \mathbf{F}_{\mathrm{std}(u)} \otimes \mathbf{F}_{\mathrm{std}(v)}, \tag{2.12}$$

where $A \oplus B$ is the ordinal sum of two mutually commuting totally ordered alphabets.

This is a polynomial realization of the Malvenuto-Reutenauer Hopf algebra $\mathbb{K}[\mathfrak{S}]$ [8].

The Hopf algebra **FQSym** is self-dual and the dual basis of $\mathbf{F}_\sigma$ is $\mathbf{G}_\sigma$ where $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$. The convolution product of two permutations is obtained by the product of two polynomials

$$\mathbf{G}_{\sigma_1} \cdot \mathbf{G}_{\sigma_2} = \sum_{\tau \in \sigma_1 * \sigma_2} \mathbf{G}_\tau. \tag{2.13}$$

3. Uniform block permutations

3.1. Definitions.

Definition 3.1. *A* uniform block permutation *of size n is a bijection between two set partitions of $[n]$ that maps each block to a block of the same cardinality.*

Let us denote by $\mathcal{UBP}_n$ the set of uniform block permutations of size n and by $\mathcal{UBP}$ the set $\cup_{n\geq 0}\mathcal{UBP}_n$. We will represent a uniform block permutation in the form of an array with two rows. Let $\mathcal{A} = \{A_1, A_2, \cdots, A_k\}$ and $\mathcal{A}' = \{A'_1, A'_2, \cdots, A'_k\}$ be two set partitions of the same size. We denote the uniform block permutation $f : \mathcal{A} \longrightarrow \mathcal{A}'$ for which the image of A_i is A'_i by

$$\left(\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_k \\ A'_1 & A'_2 & \cdots & A'_k \end{array}\right).$$

Example 3.2. Here are all uniform block permutations of size 1 and 2:

$$n=1 \quad \left(\begin{array}{c} 1 \\ 1 \end{array}\right)$$

$$n=2 \quad \left(\begin{array}{c} 12 \\ 12 \end{array}\right), \left(\begin{array}{c|c} 1 & 2 \\ 1 & 2 \end{array}\right), \left(\begin{array}{c|c} 1 & 2 \\ 2 & 1 \end{array}\right).$$

In the case where the set partitions consist of singletons (*i.e.*, $\mathcal{A} = \big\{\{1\}, \cdots, \{n\}\big\}$), we find the permutations of size n.

Definition 3.3. *Let f and g be two uniform block permutations. The* concatenation *of f and g, denoted by $f \times g$, is the uniform block permutation obtained by adding the size of f to all entries of g and concatenating the result to f.*

Example 3.4.

$$\left(\begin{array}{c|c} 13 & 2 \\ 23 & 1 \end{array}\right) \times \left(\begin{array}{c|c|c|c} 13 & 2 & 46 & 5 \\ 46 & 2 & 15 & 3 \end{array}\right) = \left(\begin{array}{c|c|c|c|c|c} 13 & 2 & 46 & 5 & 79 & 8 \\ 23 & 1 & 79 & 5 & 48 & 6 \end{array}\right).$$

Definition 3.5. *Let f and g be two uniform block permutations of the same size n, $f : \mathcal{A} \longrightarrow \mathcal{A}'$ and $g : \mathcal{B} \longrightarrow \mathcal{B}'$. The* composition *$g \circ f : \mathcal{C} \longrightarrow \mathcal{C}'$ of f and g is defined by the following process. The blocks C of the set partition $\mathcal{C}$ are the subsets of $[n]$ which are minimal for the two properties*

(1) C is a union of blocks of $\mathcal{A}$
(2) $f(C)$ is a union of blocks B_i of $\mathcal{B}$.

The image of C is the union of images $g(B_i)$.

Note that if f is a permutation, then the set partition $\mathcal{C}'$ is $\mathcal{B}'$, and if g is a permutation, then the set partition $\mathcal{C}$ is $\mathcal{A}$.

Example 3.6.

$$\left(\begin{array}{c|c} 125 & 346 \\ 236 & 145 \end{array}\right) \circ \left(\begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 6 & 3 & 1 \end{array}\right) = \left(\begin{array}{c|c} 136 & 245 \\ 236 & 145 \end{array}\right),$$

$$\left(\begin{array}{c|c|c|c|c} 15 & 2 & 3 & 4 & 6 \\ 46 & 2 & 1 & 5 & 3 \end{array}\right) \circ \left(\begin{array}{c|c|c|c} 14 & 2 & 36 & 5 \\ 23 & 1 & 45 & 6 \end{array}\right) = \left(\begin{array}{c|c|c} 14 & 236 & 5 \\ 12 & 456 & 3 \end{array}\right).$$

3.2. **Decompositions.** The data of a permutation and a set partition is enough to describe a uniform block permutation. Given a uniform block permutation $f : \mathcal{A} \longrightarrow \mathcal{A}'$ of size n, there is a permutation σ in $\mathfrak{S}_n$ such that

$$\mathcal{A}' = \bigcup_{H \in \mathcal{A}} \Big\{ \bigcup_{x \in H} \{\sigma(x)\} \Big\}. \tag{3.1}$$

Conversely, a set partition of the set $[n]$ and a permutation of $\mathfrak{S}_n$ determine a uniform block permutation of size n. Any uniform block permutation can thus be decomposed (non-uniquely) as

$$f = \sigma \circ Id_{\mathcal{A}} = Id_{\mathcal{A}'} \circ \sigma. \tag{3.2}$$

The first equality expresses the fact that one can group the images σ_i whose indices are in the same block in $\mathcal{A}$, the second expresses the fact that one can group the images σ_i whose values are in the same block in $\mathcal{A}'$.

Example 3.7.

$$\begin{aligned} \left(\begin{array}{c|c|c|c} 13 & 2 & 458 & 67 \\ 27 & 3 & 146 & 58 \end{array}\right) &= \left(\begin{array}{c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 7 & 1 & 4 & 5 & 8 & 6 \end{array}\right) \circ \left(\begin{array}{c|c|c|c} 13 & 2 & 458 & 67 \\ 13 & 2 & 458 & 67 \end{array}\right) \\ &= \left(\begin{array}{c|c|c|c} 146 & 27 & 3 & 58 \\ 146 & 27 & 3 & 58 \end{array}\right) \circ \left(\begin{array}{c|c|c|c|c|c|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 7 & 1 & 4 & 5 & 8 & 6 \end{array}\right). \end{aligned}$$

3.3. **Algebraic structures.** In [7], Aguiar and Orellana introduced a Hopf algebra whose basis is the set of uniform block permutations. It will be denoted by **UBP** in the sequel.

3.3.1. **UBP**. The product is defined as follows.

Definition 3.8. *Let f and g be two uniform block permutations of respective sizes m and n. The* convolution, *denoted by $f * g$, is the sum of all uniform block permutations of the form*

$$\left(\begin{array}{c|c|c|c} 1 & 2 & \cdots & m+n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{m+n} \end{array}\right) \circ (f \times g) \tag{3.3}$$

where

$$\mathrm{std}(\sigma_1 \cdots \sigma_m) = 12 \cdots m \;\; \text{and} \;\; \mathrm{std}(\sigma_{m+1} \cdots \sigma_{m+n}) = 12 \cdots n$$

i.e., *σ appears in $12 \cdots m * 12 \cdots n$.*

Example 3.9.

$$\begin{aligned} \left(\begin{array}{c} 1 \\ 1 \end{array}\right) * \left(\begin{array}{c} 12 \\ 12 \end{array}\right) &= \left[\left(\begin{array}{c|c|c} 1 & 2 & 3 \\ 1 & 2 & 3 \end{array}\right) + \left(\begin{array}{c|c|c} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right) + \left(\begin{array}{c|c|c} 1 & 2 & 3 \\ 3 & 1 & 2 \end{array}\right)\right] \circ \left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right) \\ &= \left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right) + \left(\begin{array}{c|c} 1 & 23 \\ 2 & 13 \end{array}\right) + \left(\begin{array}{c|c} 1 & 23 \\ 3 & 12 \end{array}\right). \end{aligned}$$

This product can be rewritten in terms of the decomposition of uniform block permutations. Given $f : \mathcal{A} \longrightarrow \mathcal{A}'$ and $g : \mathcal{B} \longrightarrow \mathcal{B}'$ two uniform block permutations, $f = \sigma \circ Id_{\mathcal{A}}$ and $g = \tau \circ Id_{\mathcal{B}}$, we have

$$f * g = (\sigma * \tau) \circ Id_{\mathcal{A} \times \mathcal{B}}. \tag{3.4}$$

Example 3.10.

$$\begin{aligned}\begin{pmatrix} 1 \\ 1 \end{pmatrix} * \begin{pmatrix} 12 \\ 12 \end{pmatrix} &= 1 \circ Id_{\{\{1\}\}} * 12 \circ Id_{\{\{1,2\}\}} \\ &= (1 * 12) \circ Id_{\{\{1\},\{2,3\}\}} \\ &= (123 + 213 + 312) \circ Id_{\{\{1\},\{2,3\}\}} \\ &= \left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right) + \left(\begin{array}{c|c} 1 & 23 \\ 2 & 13 \end{array}\right) + \left(\begin{array}{c|c} 1 & 23 \\ 3 & 12 \end{array}\right)\end{aligned}$$

Let $\mathbb{K}\mathcal{UBP}_n$ be the $\mathbb{K}$-vector space spanned by $\mathcal{UBP}_n$ and

$$\mathbf{UBP} = \bigoplus_{n \geq 0} \mathbb{K}\mathcal{UBP}_n. \tag{3.5}$$

We shall denote by $\{\mathbf{G}_f ; f \in \mathcal{UBP}\}$ the basis of **UBP**. We endow **UBP** with the product $\cdot$ defined, for uniform block permutations f and g, by

$$\mathbf{G}_f \cdot \mathbf{G}_g = \sum_{h \in f * g} \mathbf{G}_h. \tag{3.6}$$

Given a set partition $\mathcal{C}$, we set

$$\mathcal{C}_{|\{1,\cdots,i\}} = \Big\{ H \in \mathcal{C} ; H \subset [i] \Big\} \qquad \mathcal{C}_{|\{i+1,\cdots,n\}} = \operatorname{std}\left(\mathcal{C} - \mathcal{C}_{|\{1,\cdots,i\}}\right). \tag{3.7}$$

Let $\mathrm{B}(\mathcal{C})$ be the set of integers i such that

$$\{1, 2, \cdots, n\} = \mathcal{C}_{|\{1,\cdots,i\}} \times \mathcal{C}_{|\{i+1,\cdots,n\}}. \tag{3.8}$$

Example 3.11. *Let* $\mathcal{C} = \big\{\{1,3\},\{2\},\{4,6,7\},\{5\}\big\}$. *Then,*

$$\mathcal{C}_{\{1,\cdots,2\}} = \big\{\{2\}\big\} \qquad \mathcal{C}_{\{3,\cdots,7\}} = \big\{\{1,3,4\},\{2\}\big\}$$
$$\mathcal{C}_{\{1,\cdots,3\}} = \big\{\{1,3\},\{2\}\big\} \qquad \mathcal{C}_{\{4,\cdots,7\}} = \big\{\{1,3,4\},\{2\}\big\}$$

$$\mathrm{B}(\mathcal{C}) = \{0, 3, 7\}$$

Definition 3.12. *Let f be a uniform block permutation of size n, $f : \mathcal{A} \longrightarrow \mathcal{A}'$. By Lemma (2.2) in [7], i is in $\mathrm{B}(\mathcal{A}')$ if and only if there exists a unique permutation σ in $\mathfrak{S}_n$ appearing in the convolution product $12 \cdots i * 12 \cdots n - i$ and unique uniform block permutations $f_{(i)}$ of size i and $f'_{(n-i)}$ of size $n - i$ such that*

$$f = (f_{(i)} \times f'_{(n-i)}) \circ \left(\begin{array}{c|c|c|c} 1 & 2 & \cdots & n \\ (\sigma^{-1})_1 & (\sigma^{-1})_2 & \cdots & (\sigma^{-1})_n \end{array}\right) \tag{3.9}$$

The coproduct is then

$$\Delta(\mathbf{G}_f) := \sum_{i \in \mathrm{B}(\mathcal{A}')} \mathbf{G}_{f_{(i)}} \otimes \mathbf{G}_{f'_{(n-i)}}. \tag{3.10}$$

Example 3.13.

$$\begin{aligned}\Delta\mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)} &= 1 \otimes \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} \otimes \mathbf{G}_{\left(\begin{array}{c|c} 13 & 2 \\ 12 & 3 \end{array}\right)} \\ &+ \mathbf{G}_{\left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right)} \otimes \mathbf{G}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)} \otimes 1\end{aligned}$$

since

$$\begin{aligned}\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right) &= \left[(\ \ ) \times \left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)\right] \circ \left(\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}\right) \\ &= \left[\left(\begin{array}{c} 1 \\ 1 \end{array}\right) \times \left(\begin{array}{c|c} 13 & 2 \\ 12 & 3 \end{array}\right)\right] \circ \left(\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}\right) \\ &= \left[\left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right) \times \left(\begin{array}{c} 1 \\ 1 \end{array}\right)\right] \circ \left(\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{array}\right) \\ &= \left[\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right) \times (\ \ )\right] \circ \left(\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}\right).\end{aligned}$$

This coproduct can also be rewritten using the decomposition of uniform block permutations. Given $f : \mathcal{A} \longrightarrow \mathcal{A}'$ and a decomposition $f = Id_{\mathcal{A}'} \circ \sigma$, we have

$$\Delta(\mathbf{G}_f) = \sum_{i \in \mathrm{B}(\mathcal{A}')} \mathbf{G}_{f_{|\{1,\cdots,i\}}} \otimes \mathbf{G}_{f_{|\{i+1,\cdots,n\}}} \tag{3.11}$$

where

$$f_{|\{1,\cdots,i\}} = Id_{\mathcal{A}'_{|\{1,\cdots,i\}}} \circ \sigma_{|\{1,\cdots,i\}} \tag{3.12}$$

$$f_{|\{i+1,\cdots,n\}} = Id_{\mathcal{A}'_{|\{i+1,\cdots,n\}}} \circ \mathrm{std}(\sigma_{|\{i+1,\cdots,n\}}). \tag{3.13}$$

Example 3.14. We set $\mathcal{A} = \big\{\{1\},\{2,3\},\{4\}\big\}$:

$$\begin{aligned}\Delta\mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)} =& \Delta\mathbf{G}_{Id_{\mathcal{A}}\circ 1243} \\ =& 1 \otimes \mathbf{G}_{Id_{\mathcal{A}}\circ 1243} + \mathbf{G}_{Id_{\{\{1\}\}}\circ 1} \otimes \mathbf{G}_{Id_{\{\{1,2\},\{3\}\}}\circ 132} \\ &+ \mathbf{G}_{Id_{\{\{1\},\{2,3\}\}}\circ 123} \otimes \mathbf{G}_{Id_{\{\{1\}\}}\circ 1} + \mathbf{G}_{Id_{\mathcal{A}}\circ 1243} \otimes 1 \\ =& 1 \otimes \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} \otimes \mathbf{G}_{\left(\begin{array}{c|c} 13 & 2 \\ 12 & 3 \end{array}\right)} \\ &+ \mathbf{G}_{\left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right)} \otimes \mathbf{G}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 1 & 23 & 4 \end{array}\right)} \otimes 1.\end{aligned}$$

3.3.2. $\mathbf{UBP}^*$. It is proved in [7] that **UBP** is a self-dual Hopf algebra. We can describe the dual structure. Let $\mathbf{F}_f$ be the basis of $\mathbf{UBP}^*$ dual to $\mathbf{G}_f$.

The product and coproduct are, for $f : \mathcal{A} \longrightarrow \mathcal{A}'$ and $g : \mathcal{B} \longrightarrow \mathcal{B}'$ two uniform block permutations with decompositions $f = Id_{\mathcal{A}'} \circ \sigma$ and $g = Id_{\mathcal{B}'} \circ \tau$

$$\mathbf{F}_f \cdot \mathbf{F}_g = \sum_{h \in f \,\sqcup\!\sqcup\, g} \mathbf{F}_h \tag{3.14}$$

where

$$f \sqcup\!\sqcup\, g = Id_{\mathcal{A}' \times \mathcal{B}'} \circ (\sigma \sqcup\!\sqcup\, \tau) \tag{3.15}$$

and, for $f : \mathcal{A} \longrightarrow \mathcal{A}'$, with $f = \sigma \circ Id_{\mathcal{A}}$

$$\Delta\mathbf{F}_f = \sum_{i \in \mathrm{B}(\mathcal{A})} \mathbf{F}_{f_1 \cdots f_i} \otimes \mathbf{F}_{f_{i+1}\cdots f_n} \tag{3.16}$$

where

$$\begin{aligned} f_1 \cdots f_i &= \mathrm{std}(\sigma_1 \cdots \sigma_i) \circ Id_{\mathcal{A}_{|\{1,\cdots,i\}}} \\ f_{i+1} \cdots f_n &= \mathrm{std}(\sigma_{i+1} \cdots \sigma_n) \circ Id_{\mathcal{A}_{|\{i+1,\cdots,n\}}}. \end{aligned} \tag{3.17}$$

Example 3.15.

$$\mathbf{F}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} \cdot \mathbf{F}_{\left(\begin{array}{c} 12 \\ 12 \end{array}\right)} = \mathbf{F}_{\left(\begin{array}{c|c} 1 & 23 \\ 1 & 23 \end{array}\right)} + \mathbf{F}_{\left(\begin{array}{c|c} 2 & 13 \\ 1 & 23 \end{array}\right)} + \mathbf{F}_{\left(\begin{array}{c|c} 3 & 12 \\ 1 & 23 \end{array}\right)}$$

We set $\mathcal{A} = \big\{\{1\},\{2,4\},\{3\}\big\}$. Then, we have

$$\begin{aligned}\Delta\mathbf{F}_{\left(\begin{smallmatrix}1 & 24 & 3\\ 1 & 23 & 4\end{smallmatrix}\right)} =& \Delta\mathbf{F}_{1243\circ Id_{\mathcal{A}}}\\ =& 1\otimes\mathbf{F}_{1243\circ Id_{\mathcal{A}}} + \mathbf{F}_{1\circ Id_{\{\{1\}\}}}\otimes\mathbf{F}_{132\circ Id_{\{\{1,3\},\{2\}\}}} + \mathbf{F}_{1243\circ Id_{\mathcal{A}}}\otimes 1\\ =& 1\otimes\mathbf{F}_{\left(\begin{smallmatrix}1 & 24 & 3\\ 1 & 23 & 4\end{smallmatrix}\right)} + \mathbf{F}_{\left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)}\otimes\mathbf{F}_{\left(\begin{smallmatrix}13 & 2\\ 12 & 3\end{smallmatrix}\right)} + \mathbf{F}_{\left(\begin{smallmatrix}1 & 24 & 3\\ 1 & 23 & 4\end{smallmatrix}\right)}\otimes 1\end{aligned}$$

Setting $\mathbf{G}_f = \mathbf{F}_{f^{-1}}$, we can identify **UBP** and **UBP***. If $\langle\,,\,\rangle$ denotes the scalar product such that $\langle\mathbf{G}_f,\mathbf{F}_g\rangle = \delta_{fg}$, we have

$$\langle\Delta\mathbf{G}_f,\mathbf{F}_g\otimes\mathbf{F}_h\rangle = \langle\mathbf{G}_f,\mathbf{F}_g\cdot\mathbf{F}_h\rangle \tag{3.18}$$

$$\langle\mathbf{G}_f\cdot\mathbf{G}_g,\mathbf{F}_h\rangle = \langle\mathbf{G}_f\otimes\mathbf{G}_g,\Delta\mathbf{F}_h\rangle, \tag{3.19}$$

which shows that **UBP** is indeed self-dual.

4. Polynomial realization

Given two alphabets $B = \big\{b_i; 0\le i\big\}$ and $C = \{c_i; 0\le i\}$, with C totally ordered, we consider an alphabet $A = \left\{\begin{pmatrix}b_i\\ c_j\end{pmatrix}; 0\le i,j\right\}$ of bi-letters endowed with two relations

- for all i and k, $\begin{pmatrix}b_i\\ c_j\end{pmatrix}\preceq\begin{pmatrix}b_k\\ c_l\end{pmatrix}$ if and only if $c_j\le c_l$,
- for all j and l, $\begin{pmatrix}b_i\\ c_j\end{pmatrix}\equiv\begin{pmatrix}b_k\\ c_l\end{pmatrix}$ if and only if $b_i = b_k$.

The product of two biwords is the concatenation

$$\begin{pmatrix}u_1\\ v_1\end{pmatrix}\cdot\begin{pmatrix}u_2\\ v_2\end{pmatrix} = \begin{pmatrix}u_1\cdot u_2\\ v_1\cdot v_2\end{pmatrix}. \tag{4.1}$$

We extend the definition of the standardization to biwords. Let $w = \begin{pmatrix}u\\ v\end{pmatrix}$ be a biword. The *standardized* of w is the permutation $\mathrm{std}(v)$.

Example 4.1.

$$\mathrm{std}\left[\begin{pmatrix}b_1b_1b_3b_1b_2\\ c_2c_3c_2c_1c_6\end{pmatrix}\right] = 24315.$$

For any uniform block permutation $f:\mathcal{A}\longrightarrow\mathcal{A}'$, we denote by ξ_f the minimum permutation (for the lexicographic order) such that

$$f = \xi_f\circ Id_{\mathcal{A}}.$$

We say that a biword w is f-compatible, and we write $w\vdash f$, if

- $\mathrm{std}(w) = \xi_f$

and

- if i and j are in the same block in $\mathcal{A}$, then the bi-letters w_i and w_j satisfy $w_i\equiv w_j$.

We define

$$\mathbf{G}_f(A) = \sum_{w:w\vdash f} w. \tag{4.2}$$

Example 4.2. *Let* $f = \left(\begin{array}{c|c|c}13 & 256 & 4\\ 15 & 234 & 6\end{array}\right)$. *The permutation* $\sigma = 125634$ *is the minimum permutation associated with* f *and we have*

$$\mathbf{G}_f(A) = \sum_{\substack{i,j,k\\ \mathrm{std}(c_1c_2\cdots c_6)=\sigma}}\begin{pmatrix}b_ib_jb_ib_kb_jb_j\\ c_1c_2c_3c_4c_5c_6\end{pmatrix}.$$

Theorem 4.3. *The polynomials* $\mathbf{G}_f(A)$ *provide a realization of the Hopf algebra* **UBP**. *That is,*

$$\mathbf{G}_f(A)\cdot\mathbf{G}_g(A)=\sum_{h\in f*g}\mathbf{G}_h(A) \tag{4.3}$$

and

$$\mathbf{G}_f(A+A')=\Delta(\mathbf{G}_f), \tag{4.4}$$

where $A+A'$ *is defined as follows.* A' *is an alphabet isomorphic to* A *such that, for any bi-letter* w' *of* A' *and for any bi-letter* w *of* A*, we have*

$$w\preceq w'.$$

Then, $A+A'$ *denotes the disjoint union of* A *and* A' *endowed with the relations* $\preceq$ *and* $\equiv$*. As usual, we allow the bi-letters of* A *and* A' *to commute and identify* $P(A)\,Q(A')$ *with* $P\otimes Q$*.*

Proof. Let us first prove that

$$\mathbf{G}_f(A)\cdot\mathbf{G}_g(A)=\sum_{h\in f*g}\mathbf{G}_h(A).$$

Writing $f=\xi_f\circ Id_{\mathcal{A}}$ and $g=\xi_g\circ Id_{\mathcal{B}}$, we have

$$\begin{aligned}\mathbf{G}_f(A)\cdot\mathbf{G}_g(A)&=\sum_{w:w\vdash f}w\quad\sum_{w:w\vdash g}w\\&=\sum_{\substack{u_1\in B^*,v_1\in C^*\\ u_1\text{ is }\mathcal{A}\text{-compatible}\\ \mathrm{std}(v_1)=\xi_f}}\begin{pmatrix}u_1\\v_1\end{pmatrix}\sum_{\substack{u_2\in B^*,v_2\in C^*\\ u_2\text{ is }\mathcal{B}\text{-compatible}\\ \mathrm{std}(v_2)=\xi_g}}\begin{pmatrix}u_2\\v_2\end{pmatrix}\\&=\sum_{\sigma\in\xi_f*\xi_g}\ \sum_{\substack{u\in B^*,v\in C^*\\ u\text{ is }\mathcal{A}\times\mathcal{B}\text{-compatible}\\ \mathrm{std}(v)=\sigma}}\begin{pmatrix}u\\v\end{pmatrix}.\end{aligned}$$

But, given two decompositions $f=\sigma\circ Id_{\mathcal{A}}$ and $g=\tau\circ Id_{\mathcal{B}}$, if σ (*resp.* τ) is the minimum permutation associated with f (*resp.* g), then each permutation ν occuring in the product $\sigma*\tau$ is the minimum permutation associated with $\nu\circ Id_{\mathcal{A}\times\mathcal{B}}$.

Let us now show the second point. Let w be a biword occuring in $\mathbf{G}_f(A+A')$. Let i be the number of bi-letters of w from the alphabet A and let $\begin{pmatrix}u_1u_2\cdots u_i\\v_1v_2\cdots v_i\end{pmatrix}\in A^*$ be the sub-biword of w on these letters. Then

$$\mathrm{std}(v_1v_2\cdots v_i)=(\xi_f)_{|\{1,2,\cdots,i\}}$$

and

$$u_1u_2\cdots u_i\ \text{ is }\ \mathrm{std}(\mathcal{A}'_{|\{\xi_f^{-1}(1),\xi_f^{-1}(2),\cdots,\xi_f^{-1}(i)\}})\text{-compatible.}$$

This partition and the permutation $(\xi_f)_{|\{1,2,\cdots,i\}}$ define a uniform block permutation g whose minimum permutation is $(\xi_f)_{|\{1,2,\cdots,i\}}$. Hence, the biword $\begin{pmatrix}u_1u_2\cdots u_i\\v_1v_2\cdots v_i\end{pmatrix}$ is a term appearing in $\mathbf{G}_g(A)$. Similarly, the biword w defined on the alphabet A' appears in $\mathbf{G}_h(A')$ for some uniform block permutation h. By construction, $\mathbf{G}_g\otimes\mathbf{G}_h$ is a term appearing in $\Delta(\mathbf{G}_f)$.

Conversely, let $P\otimes Q$ be a term occuring in $\Delta(\mathbf{G}_f)$. Let $w=\begin{pmatrix}u\\v\end{pmatrix}$ be the biword in $P(A)\sqcup Q(A')$ such that $\mathrm{std}(w)=\xi_f$. The biword u is $\mathcal{A}'_{|\{\xi_f^{-1}(1),\xi_f^{-1}(2),\cdots,\xi_f^{-1}(n)\}}$-compatible. Thus u is $\mathcal{A}$-compatible and w appears in $\mathbf{G}_f(A+A')$.

It remains to show that the family $\mathbf{G}_f(\mathcal{A})$ is independent. To this aim, we need an order relation on the set partitions of $[n]$. Let $\mathcal{A} = \{A_1, \cdots, A_k\}$ and $\mathcal{B} = \{B_1, \cdots, B_l\}$ be two set partitions. We say that $\mathcal{B}$ is finer than $\mathcal{A}$, and we write $\mathcal{A} \preceq \mathcal{B}$, if and only if

$$(\forall 1 \leq i \leq k)\,(\exists H \subset \{1, \cdots, l\}),\ A_i = \bigcup_{j \in H} B_j.$$

Let $u \in B^*$ be a word. We set

$$\mathcal{P}(u) = \bigcup_{a \in \mathrm{Alph}(u)} \big\{\{i; u_i = a\}\big\}. \tag{4.5}$$

For example,

$$\mathcal{P}(121212) = \{\{1,3,5\},\{2,4,6\}\} \qquad \mathcal{P}(12\cdots n) = \{\{1\},\{2\},\cdots,\{n\}\}.$$

We define on the set of biwords the following relation. Given two biwords $w_1, w_2 \in A^*$, $w_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$:

$$w_1 \trianglelefteq w_2 \Longleftrightarrow \mathcal{P}(u_1) \preceq \mathcal{P}(u_2) \text{ and } \mathrm{std}(v_1) = \mathrm{std}(v_2).$$

Let f be a uniform block permutation. We denote by w_f a maximal f-compatible biword for the relation $\trianglelefteq$. We define

$$\tilde{G}_f(A) = \sum_{\substack{w \in A^* \\ \mathrm{std}(w) = \xi_f \\ \mathcal{P}(w_f) = \mathcal{P}(w)}} w. \tag{4.6}$$

The family $\big\{\tilde{\mathbf{G}}_f(A)\big\}_f$ is independent since any biword $w = \begin{pmatrix} u \\ v \end{pmatrix}$ appearing in $\tilde{\mathbf{G}}_f(A)$ allows to reconstruct f:

$$f = \mathrm{std}(v) \circ Id_{\mathcal{P}(u)}. \tag{4.7}$$

But

$$\begin{aligned} \mathbf{G}_f(A) &= \sum_{w \in A^* : w \trianglelefteq w_f} w \\ &= \sum_{\mathcal{B} \in \mathcal{P} : \mathcal{B} \preceq \mathcal{P}(w_f)} \ \sum_{\substack{w \in A^* \\ \mathcal{P}(w) = \mathcal{B} \\ \mathrm{std}(w) = \xi_f}} w. \end{aligned}$$

The partition $\mathcal{B} \in \mathcal{P}$ and the permutation $\xi_f \in \mathfrak{S}$ define a uniform block permutation g. Thus, we have

$$\tilde{\mathbf{G}}_g(A) = \sum_{\substack{w \in A^* \\ \mathcal{P}(w) = \mathcal{B} \\ \mathrm{std}(w) = \xi_f}} w.$$

Hence

$$\mathbf{G}_f(A) = \sum_{\substack{\mathcal{B} \preceq \mathcal{P}(w_f) \\ g = \xi_f \circ Id_{\mathcal{B}}}} \tilde{\mathbf{G}}_g(A).$$

Since the family $\big\{\tilde{\mathbf{G}}_g(A)\big\}$ is independent, so is the family $\big\{\mathbf{G}_f(A)\big\}$. $\square$

Example 4.4.

$$
\begin{aligned}
\mathbf{G}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}(A)\cdot\mathbf{G}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}(A) &= \sum_{\substack{b\in B\\ c\in C}}\binom{b}{c}^2 = \sum_{\substack{b_1,b_2\in B\\ c_1,c_2\in C}}\binom{b_1b_2}{c_1c_2}\\
&= \sum_{\substack{b_1,b_2\in B\\ c_1,c_2\in C: c_1\le c_2}}\binom{b_1b_2}{c_1c_2} + \sum_{\substack{b_1,b_2\in B\\ c_1,c_2\in C: c_1> c_2}}\binom{b_1b_2}{c_1c_2}\\
&= \mathbf{G}_{\left(\begin{smallmatrix}1&|&2\\1&|&2\end{smallmatrix}\right)}(A) + \mathbf{G}_{\left(\begin{smallmatrix}1&|&2\\2&|&1\end{smallmatrix}\right)}(A).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G}_{\left(\begin{smallmatrix}12\\12\end{smallmatrix}\right)}(A)\cdot\mathbf{G}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}(A) &= \sum_{\substack{b\in B\\ \mathrm{std}(v)=12}}\binom{bb}{v}\sum_{\substack{b\in B\\ c\in C}}\binom{b}{c} = \sum_{\substack{i_1,i_2\\ \mathrm{std}(v_1)=12\\ \mathrm{std}(v_2)=1}}\binom{b_{i_1}b_{i_1}b_{i_2}}{v_1v_2}\\
&= \sum_{\substack{i_1,i_2\\ c_1\le c_2\le c_3}}\binom{b_{i_1}b_{i_1}b_{i_2}}{c_1c_2c_3} + \sum_{\substack{i_1,i_2\\ c_1\le c_3< c_2}}\binom{b_{i_1}b_{i_1}b_{i_2}}{c_1c_2c_3} + \sum_{\substack{i_1,i_2\\ c_3< c_1\le c_2}}\binom{b_{i_1}b_{i_1}b_{i_2}}{c_1c_2c_3}\\
&= \mathbf{G}_{\left(\begin{smallmatrix}12&|&3\\12&|&3\end{smallmatrix}\right)} + \mathbf{G}_{\left(\begin{smallmatrix}12&|&3\\13&|&2\end{smallmatrix}\right)} + \mathbf{G}_{\left(\begin{smallmatrix}12&|&3\\23&|&1\end{smallmatrix}\right)}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G}_{\left(\begin{smallmatrix}13&|&24\\13&|&24\end{smallmatrix}\right)}(A+A') &= \sum_{\substack{i,j\\ c_1\le c_2\le c_3\le c_4}}\binom{b_ib_jb_ib_j}{c_1c_2c_3c_4} + \sum_{\substack{i,j\\ c'_1\le c'_2\le c'_3\le c'_4}}\binom{b'_ib'_jb'_ib'_j}{c'_1c'_2c'_3c'_4}\\
&= \mathbf{G}_{\left(\begin{smallmatrix}13&|&24\\13&|&24\end{smallmatrix}\right)}(A) + \mathbf{G}_{\left(\begin{smallmatrix}13&|&24\\13&|&24\end{smallmatrix}\right)}(A').
\end{aligned}
$$

We set $\tau = 2431$.

$$
\begin{aligned}
\mathbf{G}_{\left(\begin{smallmatrix}13&|&2&|&4\\23&|&4&|&1\end{smallmatrix}\right)}(A+A') &= \sum_{\substack{i,j,k\\ \mathrm{std}(c_1c_2c_3c_4)=\tau}}\binom{b_ib_jb_ib_k}{c_1c_2c_3c_4} + \sum_{\substack{i,j,k\\ \mathrm{std}(c'_1c'_2c'_3c_4)=\tau}}\binom{b'_ib'_jb'_ib_k}{c'_1c'_2c'_3c_4}\\
&\quad + \sum_{\substack{i,j,k\\ \mathrm{std}(c_1c'_2c_3c_4)=\tau}}\binom{b_ib'_jb_ib_k}{c_1c'_2c_3c_4} + \sum_{\substack{i,j,k\\ \mathrm{std}(c'_1c'_2c'_3c'_4)=\tau}}\binom{b'_ib'_jb'_ib'_k}{c'_1c'_2c'_3c'_4}\\
&= \mathbf{G}_{\left(\begin{smallmatrix}13&|&2&|&4\\23&|&4&|&1\end{smallmatrix}\right)}(A) + \mathbf{G}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}(A)\cdot\mathbf{G}_{\left(\begin{smallmatrix}13&|&2\\12&|&3\end{smallmatrix}\right)}(A')\\
&\quad + \mathbf{G}_{\left(\begin{smallmatrix}12&|&3\\23&|&1\end{smallmatrix}\right)}(A)\cdot\mathbf{G}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}(A') + \mathbf{G}_{\left(\begin{smallmatrix}13&|&2&|&4\\23&|&4&|&1\end{smallmatrix}\right)}(A').
\end{aligned}
$$

5. Packed words

We recall the definitions of packed words and of the Hopf algebra **WQSym** (see, *e.g.*, [10]). Let $A := \{a_1 < a_2 < \cdots\}$ be a totally ordered infinite alphabet. The *packed word* $\mathrm{pack}(w)$ associated with a word $w \in A^*$ is the word u obtained by the following process. If $b_1 < b_2 < \cdots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \longrightarrow a_i$. A word w is said to be *packed* if $\mathrm{pack}(w)$ is w.

Example 5.1.

$$\mathrm{pack}(51735514) = 41524413 \qquad \mathrm{pack}(772335) = 441223.$$

5.1. The combinatorial Hopf algebra $\mathbf{WQSym}^*$.

5.1.1. $\mathbf{WQSym}^*$. The Hopf algebra $\mathbf{WQSym}^*$ is the $\mathbb{K}$-vector space generated by packed words endowed with the shifted shuffle product and the coproduct given by, using notations in [5]

$$\mathbf{M}^*_u \cdot \mathbf{M}^*_v := \sum_{w \in u \,\sqcup\!\sqcup\, v[\max(u)]} \mathbf{M}^*_w, \tag{5.1}$$

$$\Delta(\mathbf{M}^*_w) := \sum_{u \cdot v = w} \mathbf{M}^*_{\mathrm{pack}(u)} \otimes \mathbf{M}^*_{\mathrm{pack}(v)}. \tag{5.2}$$

Let ψ be the map that associates with a uniform block permutation $f = Id_{\mathcal{A}'} \circ \sigma$ of size n a word w of length n constructed by packing the word u obtained by the following process. The ith letter u_i of u is the minimum of the letters σ_j which are in the same block of $\mathcal{A}$ as σ_i.

Example 5.2.

$$\psi(Id_{\{\{1,2\}\}} \circ 12) = 11 \qquad \psi(Id_{\{\{1\},\{2,3\}\}} \circ 231) = 221$$
$$\psi(Id_{\{\{1,2\},\{3,4\}\}} \circ 1324) = 1212 \qquad \psi(Id_{\{\{1,5\},\{2\},\{3\},\{4\}\}} \circ 23415) = 23411$$

The map ψ induces a linear map from the vector space $\mathbf{UBP}^*$ spanned by uniform block permutations to the one spanned by packed words $\mathbf{WQSym}^*$ defined by:

$$\Psi : \begin{array}{ccc} \mathbf{UBP}^* & \longrightarrow & \mathbf{WQSym}^* \\ \mathbf{F}_f & \longmapsto & \mathbf{M}^*_{\psi(f)} \end{array} \tag{5.3}$$

Example 5.3.

$$\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\12\end{smallmatrix}\right)}\Big) = \mathbf{M}^*_{11} \qquad \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\23\end{smallmatrix}\middle|\begin{smallmatrix}3\\1\end{smallmatrix}\right)}\Big) = \mathbf{M}^*_{221}$$
$$\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}13\\12\end{smallmatrix}\middle|\begin{smallmatrix}24\\34\end{smallmatrix}\right)}\Big) = \mathbf{M}^*_{1212} \qquad \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}2\\3\end{smallmatrix}\middle|\begin{smallmatrix}3\\4\end{smallmatrix}\middle|\begin{smallmatrix}45\\15\end{smallmatrix}\right)}\Big) = \mathbf{M}^*_{23411}$$

Proposition 5.4. *Let f and g be two uniform block permutations. Then*

$$\Psi(\mathbf{F}_f \cdot \mathbf{F}_g) = \Psi(\mathbf{F}_f) \cdot \Psi(\mathbf{F}_g). \tag{5.4}$$

Proof. Let $\mathbf{M}^*_w$ be a term in the left-hand side of (5.4) and $\mathbf{M}^*_w = \Psi(\mathbf{F}_h)$ for some uniform block permutation h in the product $f \,\overline{\sqcup\!\sqcup}\, g$. The ith letter of w is obtained from a permutation σ and comes from a subset E of $\mathcal{A} \times \mathcal{B}$ containing σ_i. But, if w_i is less than the number of blocks in $\mathcal{A}$, then E is contained in $\mathcal{A}$. So the word extracted from w whose values are less than the number of blocks in $\mathcal{A}$ is $\Psi(\mathbf{F}_f)$. Similarly, the packed word extracted from w whose values are strictly greater than the number of blocks in $\mathcal{A}$ is $\Psi(\mathbf{F}_g)$. Hence, each term in the left-hand side of (5.4) is in the right-hand side.

Since the number of terms and the multiplicities are the same, we have equality. □

Example 5.5.

$$\begin{aligned}\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\13\end{smallmatrix}\middle|\begin{smallmatrix}3\\2\end{smallmatrix}\right)} \cdot \mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}2\\2\end{smallmatrix}\right)}\Big) =& \mathbf{M}^*_{11234} + \mathbf{M}^*_{11324} + \mathbf{M}^*_{11342} + \mathbf{M}^*_{13124} + \mathbf{M}^*_{13142} \\ &+ \mathbf{M}^*_{13412} + \mathbf{M}^*_{31124} + \mathbf{M}^*_{31142} + \mathbf{M}^*_{31412} + \mathbf{M}^*_{34112} \\ =& \mathbf{M}^*_{112} \cdot \mathbf{M}^*_{12} \\ =& \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\13\end{smallmatrix}\middle|\begin{smallmatrix}3\\2\end{smallmatrix}\right)}\Big) \cdot \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\middle|\begin{smallmatrix}2\\2\end{smallmatrix}\right)}\Big).\end{aligned}$$

The map Ψ cannot be a morphism of coalgebras as shown in the example

$$\begin{aligned}\Psi\otimes\Psi\circ\Delta\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\23\end{smallmatrix}\middle|\begin{smallmatrix}3\\1\end{smallmatrix}\right)}\Big) =&\Psi\otimes\Psi\Big(1\otimes\mathbf{F}_{\left(\begin{smallmatrix}12\\23\end{smallmatrix}\middle|\begin{smallmatrix}3\\1\end{smallmatrix}\right)}+\mathbf{F}_{\left(\begin{smallmatrix}12\\12\end{smallmatrix}\right)}\otimes\mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}+\mathbf{F}_{\left(\begin{smallmatrix}12\\23\end{smallmatrix}\middle|\begin{smallmatrix}3\\1\end{smallmatrix}\right)}\otimes 1\Big)\\ =&1\otimes\mathbf{M}^*_{221}+\mathbf{M}^*_{11}\otimes\mathbf{M}^*_{1}+\mathbf{M}^*_{221}\otimes 1\\ \neq&1\otimes\mathbf{M}^*_{221}+\mathbf{M}^*_{1}\otimes\mathbf{M}^*_{21}+\mathbf{M}^*_{11}\otimes\mathbf{M}^*_{1}+\mathbf{M}^*_{221}\otimes 1\\ \neq&\Delta(\mathbf{M}^*_{221}).\end{aligned}$$

The idea is that the coproduct on packed words deconcatenates the word in all suffixes and prefixes, whereas the coproduct on uniform block permutations preserves the blocks. This suggests to define an another coproduct

$$\Delta'(\mathbf{M}^*_w) := \sum_{\substack{u\cdot v=w\\ \mathrm{Alph}(u)\cap\mathrm{Alph}(v)=\emptyset}} \mathbf{M}^*_{\mathrm{pack}(u)}\otimes\mathbf{M}^*_{\mathrm{pack}(v)}.$$

Example 5.6.

$$\begin{aligned}\Delta'(\mathbf{M}^*_{221}) =&1\otimes\mathbf{M}^*_{221}+\mathbf{M}^*_{11}\otimes\mathbf{M}^*_{1}+\mathbf{M}^*_{221}\otimes 1\\ \Delta'(\mathbf{M}^*_{331412}) =&1\otimes\mathbf{M}^*_{331412}+\mathbf{M}^*_{11}\otimes\mathbf{M}^*_{1312}+\mathbf{M}^*_{22131}\otimes\mathbf{M}^*_{1}+\mathbf{M}^*_{331412}\otimes 1\end{aligned}$$

5.1.2. **WQSym**$^{*\prime}$. We introduce then a combinatorial Hopf algebra based on packed words, endowed with the product $\cdot$ and with the coproduct Δ', which will be denoted by **WQSym**$^{*\prime}$.

Indeed, the coproduct Δ' is coassociative and, by (2.2), is compatible with the shifted shuffle. Indeed, let u and v be two packed words of respective length m and n. Let w be a packed word in the product $u \,\overline{\sqcup\!\sqcup}\, v$. We shall denote by $I(w)$ the set

$$\big\{0\le i\le m+n;\mathrm{Alph}(w_1\cdots w_i)\cap\mathrm{Alph}(w_{i+1}\cdots w_{m+n})=\emptyset\big\}.$$

Since the letters appearing in u and $v[|\max(u)|]$ are distinct, if k is in $I(w)$, then there exists i in $I(u)$ and j in $I(v)$ whose sum is k. Hence

$$\begin{aligned}\Delta'(\mathbf{M}_u\cdot\mathbf{M}_v) =& \sum_{\substack{w\in u\,\overline{\sqcup\!\sqcup}\,v\\ k\in I(w)}}\mathbf{M}_{\mathrm{pack}(w_1\cdots w_k)}\otimes\mathbf{M}_{\mathrm{pack}(w_{k+1}\cdots w_{m+n})}\\ =&\sum_{\substack{0\le k\le m+n\\ w\in u\,\overline{\sqcup\!\sqcup}\,v:k\in I(w)}}\mathbf{M}_{\mathrm{pack}(w_1\cdots w_k)}\otimes\mathbf{M}_{\mathrm{pack}(w_{k+1}\cdots w_{m+n})}\\ =&\sum_{\substack{0\le k\le m+n\\ i\in I(u),j\in I(v):i+j=k\\ w\in u_1\cdots u_i\,\sqcup\!\sqcup\,v_1[m]\cdots v_j[m]\\ w'\in u_{i+1}\cdots u_m\,\sqcup\!\sqcup\,v_{j+1}[m]\cdots v_n[m]}}\mathbf{M}_{\mathrm{pack}(w)}\otimes\mathbf{M}_{\mathrm{pack}(w')}\\ =&\Delta'(\mathbf{M}_u)\cdot\Delta'(\mathbf{M}_v).\end{aligned}$$

This Hopf algebra **WQSym**$^{*\prime}$ is isomorphic to **WQSym***. This will be shown later. To this end, we introduce the following relation on packed words which will allow us to construct an isomorphism.

We denote, given a packed word u and an integer $1\le i<\max(u)$, by $f(u,i)$ the packed word v defined by

$$v_j := \begin{cases} u_j-1 & \text{if } u_j>i\\ u_j & \text{otherwise}\end{cases}$$

Example 5.7.

$$f(121,1)=111 \qquad f(3122,2)=2111$$
$$f(11122344,2)=11122233 \qquad f(42315,1)=31214$$

We extend the definition of iterated images by f by setting, given a packed word u and a subset $I = \{i_1 > i_2 > \cdots > i_k\}$ of $[\max(u)]$:

$$f(u, I) := \begin{cases} u & \text{if } I = \emptyset \\ f(\cdots f(u, i_1), \cdots, i_k) & \text{otherwise.} \end{cases}$$

We consider the relation, denoted by $\preceq$, on packed words of the same size with the cover relation, given two packed words u and v:

$$u \preceq v \iff \begin{cases} \operatorname{std}(u) = \operatorname{std}(v) \\ \text{there exists } i \text{ such that } u = f(v, i) \end{cases} \tag{5.5}$$

Example 5.8. *Here is the Hasse diagram of packed words of size* 3*:*

$$\begin{tikzcd}[column sep=small] & 111 \arrow[dl, dash] \arrow[dr, dash] & & 121 \arrow[d, dash] & 212 \arrow[d, dash] & 221 \arrow[d, dash] & 211 \arrow[d, dash] & 321 \\ 112 \arrow[dr, dash] & & 122 \arrow[dl, dash] & 132 & 213 & 231 & 312 & \\ & 123 & & & & & & \end{tikzcd}$$

Let Γ defined by

$$\Gamma : \begin{array}{ccc} \mathbf{WQSym}^* & \longrightarrow & \mathbf{WQSym}^{*\prime} \\ \mathbf{M}^*_w & \longmapsto & \displaystyle\sum_{u \succeq w} \mathbf{M}^*_u \end{array} .$$

Example 5.9. *The image of a permutation is a permutation, the image of a word of length n consisting only of* 1*s is the sum of all non-decreasing packed words of length n and*

$$\Gamma(\mathbf{M}^*_{2441235}) = \mathbf{M}^*_{2441235} + \mathbf{M}^*_{2551346} + \mathbf{M}^*_{2451236} + \mathbf{M}^*_{2561347}.$$

Proposition 5.10. *The linear map Γ is an isomorphism of Hopf algebras from* $\mathbf{WQSym}^*$ *to* $\mathbf{WQSym}^{*\prime}$*.*

Proof. To show that Γ is a morphism of algebras, that is

$$\Gamma(\mathbf{M}^*_w \cdot \mathbf{M}^*_{w'}) = \Gamma(\mathbf{M}^*_w) \cdot \Gamma(\mathbf{M}^*_{w'}), \tag{5.6}$$

we shall show that every packed word in the left-hand side of (5.6) is in the right-hand side. Since these sums are multiplicity free, it suffices to show the equality of the underlying sets. Consider two packed words w and w'. Let $u \succeq w$ and $u' \succeq w'$. Then the set $u \,\sqcup\!\sqcup\, u'$ is equal to the set of the packed words $v \succeq w \,\sqcup\!\sqcup\, w'$ such that

$$v_{|\{1,\cdots,\max(u)\}} = u \qquad \operatorname{pack}(v_{|\{\max(u)+1,\cdots,\max(v)\}}) = u'. \tag{5.7}$$

Thus, the set $\Gamma(\mathbf{M}^*_w) \cdot \Gamma(\mathbf{M}^*_{w'})$ is equal to $\Gamma(\mathbf{M}^*_w \cdot \mathbf{M}^*_{w'})$.

Let us now prove that Γ is a morphism of coalgebras, that is to say

$$\Gamma \otimes \Gamma \circ \Delta(\mathbf{M}^*_w) = \Delta' \circ \Gamma(\mathbf{M}^*_w). \tag{5.8}$$

To do this, we shall show that every packed word in the left-hand side of (5.8) is in the right-hand side and conversely. We shall conclude with an argument of multiplicity.

We consider a packed word w. We set

$$E_1 := \bigcup_{u \cdot v = w} \{u' \otimes v' ; u' \succeq \operatorname{pack}(u), v' \succeq \operatorname{pack}(v)\},$$

$$E_2 := \bigcup_{\substack{w' \succeq w \\ u \cdot v = w' \\ \operatorname{Alph}(u) \cap \operatorname{Alph}(v) = \emptyset}} \{u' \otimes v' ; \operatorname{pack}(u) = u', \operatorname{pack}(v) = v'\}$$

so that

$$\Gamma\otimes\Gamma\circ\Delta(\mathbf{M}^*_w) = \sum_{u'\otimes v'\in E_1} \mathbf{M}^*_{u'}\otimes\mathbf{M}^*_{v'}$$
$$\Delta'\circ\Gamma(\mathbf{M}^*_w) = \sum_{u'\otimes v'\in E_2} \mathbf{M}^*_{u'}\otimes\mathbf{M}^*_{v'}.$$

Show $E_2 \subset E_1$. Define $F(w,n)$ as the word u given by $u_i = w_i$ if $w_i \leq n$ and $u_i = w_i - 1$ otherwise. Let $u'\otimes v'$ in E_2. There exists $w \preceq w'$ and $w' = u\cdot v$ with $\mathrm{pack}(u) = u'$ and $\mathrm{pack}(v) = v'$. Then there exists I such that

$$w = f(w', I) = F(w', I) = F(u, I)\cdot F(v, I).$$

But, there exists I_1 and I_2 (roughly, I_1 and I_2 are a relabelling of I caused by the packing) such that

$$\mathrm{pack}(F(u,I)) = f(\mathrm{pack}(u), I_1) \qquad \mathrm{pack}(F(v,I)) = f(\mathrm{pack}(v), I_2).$$

In particular, $\mathrm{pack}(F(u,I)) \preceq u'$ and $\mathrm{pack}(F(v,I)) \preceq v'$. Hence, $u'\otimes v'$ is in E_1.

Show $E_1 \subset E_2$. Let $u'\otimes v'$ in E_2. There exists two words u and v such that $w = u\cdot v$ and $u' \succeq \mathrm{pack}(u)$, $v' \succeq \mathrm{pack}(v)$. It remains to be seen that there exists two words U and V such that

- $\mathrm{pack}(U) = u'$, $\mathrm{pack}(V) = v'$,
- $\mathrm{Alph}(U)\cap\mathrm{Alph}(V) = \emptyset$,
- $U\cdot V \succeq w$.

We illustrate the algorithm step by step to find U and V on the following example. Let $u = 311434141$, $v = 22441144$ and $w = u\cdot v$. Let $u' = 411546263$ and $v' = 23451156$ be two packed words such that $\mathrm{pack}(u) \preceq u'$ and $\mathrm{pack}(v) \preceq v'$.
The first step is to have two words U' and V' such that

- $\mathrm{pack}(u) = \mathrm{pack}(U') \preceq u'$, $\mathrm{pack}(v) = \mathrm{pack}(V') \preceq v'$,
- $\mathrm{Alph}(U')\cap\mathrm{Alph}(V') = \emptyset$,
- $U'\cdot V' \succeq w$.

This is iteratively obtained by the following process. Given a word u and an integer i, let $g(u,i)$ be the word v defined by

$$v_j = \begin{cases} u_j + 1 & \text{if } u_j \geq i \\ u_j & \text{otherwise} \end{cases}.$$

If i is a letter occuring in $\mathrm{Alph}(u)\cap\mathrm{Alph}(v)$, then we consider the words $\tilde{u} = g(u, i+1)$ and $\tilde{v} = g(v,i)$ and repeat this process with these words $\tilde{u}$ and $\tilde{v}$. At the end of the process, we have two words U' and V' satisfying the desired properties since g does not modify the packing. For example:

$$\begin{array}{rrl} & u\cdot v = & 311434141 \cdot 22441144 \\ i=1: & \tilde{u}\cdot\tilde{v} = & 411545151 \cdot 33552255 \\ i=5: & U'\cdot V' = & 411545151 \cdot 33662266 \end{array}$$

The second step, from the words U' and V', is to have the words U'' and V'' such that

- $\mathrm{pack}(U'') = u'$, $\mathrm{pack}(v) = \mathrm{pack}(V'') \preceq v'$,
- $\mathrm{Alph}(U'')\cap\mathrm{Alph}(V'') = \emptyset$,
- $U''\cdot V'' \succeq w$.

This is iteratively obtained by the following process. Since $\mathrm{pack}(U') \preceq u'$, there exists I such that $f(\mathrm{pack}(U'), I) = u'$. If I is empty, then $\mathrm{pack}(U') = u'$. Otherwise, let $i \in I$. By incrementing the letters i occuring in $g(U', i+1)$ at the position j such that $u'_j \neq i$, we obtain a word $\tilde{u}$ satisfying $\mathrm{pack}(U') \prec \mathrm{pack}(\tilde{u}) \preceq u'$. We set $\tilde{v} = g(V', i)$. We repeat this process with these words $\tilde{u}$ and $\tilde{v}$. At the end of the process, we have two words U'' and V'' satisfying the desired properties since g does not modify the packing. For example:

$$\begin{array}{rrl} & U' \cdot V' = & 411545151 \cdot 33662266 \\ i=1: & \tilde{u} \cdot \tilde{v} = & 511656262 \cdot 44773377 \\ i=2: & \tilde{u} \cdot \tilde{v} = & 611767273 \cdot 55884488 \\ i=7: & U'' \cdot V'' = & 611768283 \cdot 55994499 \\ \hline & u' \cdot v' = & 411546263 \cdot 23451156 \end{array}$$

The third step is similar to the second step. From the words U'' and V'', we obtain the words U and V satisfying the desired properties by referring the act on the word U' (*resp.* V') in the second step to V'' (*resp.* U''). For example:

$$\begin{array}{rrl} & U'' \cdot V'' = & 611768283 \cdot 55994499 \\ i=5: & \tilde{u} \cdot \tilde{v} = & 711879293 \cdot 56AA44AA \\ i=A: & \tilde{u} \cdot \tilde{v} = & 711879293 \cdot 56AB44BB \\ i=B: & U \cdot V = & 711879293 \cdot 56AB44BC \\ \hline & u' \cdot v' = & 411546263 \cdot 23451156 \end{array}$$

where A, B and C respectively means the integer 10, 11 and 12. This proves the equality $E_1 = E_2$.

Since both members of the equality (5.8) are multiplicity free, we have equality.

Finally, by the Möbius inversion, Γ is an isomorphism. The inverse, denoting by μ the Möbius function, is given by:

$$\Gamma^{-1} : \begin{array}{ccc} \mathbf{WQSym}^{*\prime} & \longrightarrow & \mathbf{WQSym}^{*} \\ \mathbf{M}^*_w & \longmapsto & \displaystyle\sum_{u \succeq w} \mu(\mathrm{std}(u), u)\ \mathbf{M}^*_u \end{array} .$$

□

Example 5.11.

$$\begin{aligned} \Gamma(\mathbf{M}^*_{11} \cdot \mathbf{M}^*_1) &= \Gamma(\mathbf{M}^*_{112} + \mathbf{M}^*_{121} + \mathbf{M}^*_{211}) \\ &= \mathbf{M}^*_{112} + \mathbf{M}^*_{123} + \mathbf{M}^*_{121} + \mathbf{M}^*_{132} + \mathbf{M}^*_{311} + \mathbf{M}^*_{312} \\ &= \mathbf{M}^*_{11} \cdot \mathbf{M}^*_1 + \mathbf{M}^*_{12} \cdot \mathbf{M}^*_1 \\ &= \Gamma(\mathbf{M}^*_{11}) \cdot \Gamma(\mathbf{M}^*_1) \end{aligned}$$

$$\begin{aligned} \Gamma \otimes \Gamma(\Delta(\mathbf{M}^*_{121})) &= \mathbf{M}^*_{121} \otimes 1 + \mathbf{M}^*_{132} \otimes 1 + \mathbf{M}^*_{12} \otimes \mathbf{M}^*_1 + \mathbf{M}^*_1 \otimes \mathbf{M}^*_{21} + 1 \otimes \mathbf{M}^*_{121} + 1 \otimes \mathbf{M}^*_{132} \\ &= \Delta'(\mathbf{M}^*_{121}) + \Delta'(\mathbf{M}^*_{132}) \\ &= \Delta'(\Gamma(\mathbf{M}^*_{121})). \end{aligned}$$

$$\begin{aligned} \Gamma^{-1}(\mathbf{M}^*_{111}) &= \mathbf{M}^*_{111} - \mathbf{M}^*_{112} - \mathbf{M}^*_{122} + \mathbf{M}^*_{123} \\ \Gamma^{-1}(\mathbf{M}^*_{421234}) &= \mathbf{M}^*_{421234} - \mathbf{M}^*_{521345} - \mathbf{M}^*_{421235} + \mathbf{M}^*_{521346} \end{aligned}$$

We can now return to the main result of this section, the linear map Ψ, defined in (5.3), is a surjective Hopf algebra morphism from $\mathbf{UBP}^*$ to $\mathbf{WQSym}^*$.

Proposition 5.12. *The map Ψ is a surjective Hopf algebra morphism from $\mathbf{UBP}^*$ to $\mathbf{WQSym}^{*\prime}$.*

Proof. That Ψ is a morphism of algebras follows from Proposition 5.4.

Let us show that Ψ is a morphism of coalgebras, that is

$$\Psi \otimes \Psi \circ \Delta(\mathbf{F}_f) = \Delta' \circ \Psi(\mathbf{F}_f). \tag{5.9}$$

Let f be a uniform block permutation of size n, $f : \mathcal{A} \longrightarrow \mathcal{A}'$ and let w be the word defined by

$$w_i = \min\left(\{(\xi_f)_j;\ \exists A \in \mathcal{A} \text{ containing } i \text{ and } j\}\right).$$

Let us write w as the concatenation $u \cdot v$ and $\mathrm{pack}(w)$ as the concatenation $u' \cdot v'$ such that u and u' are of the same length. The sets of letters in u and v are disjoint if and only if $|u|$ is in $I(\mathcal{A})$. Moreover, by definition of w, we have

$$\begin{aligned}\Psi(\mathbf{F}_{f_1\cdots f_{|u|}}) =&\mathbf{M}^*_{\mathrm{pack}(u)} = \mathbf{M}^*_{\mathrm{pack}(u')}\\ \Psi(\mathbf{F}_{f_{|u|+1}\cdots f_n}) =&\mathbf{M}^*_{\mathrm{pack}(v)} = \mathbf{M}^*_{\mathrm{pack}(v')}.\end{aligned}$$

Hence

$$\begin{aligned}\Delta'(\Psi(\mathbf{F}_f)) =& \sum_{\substack{u'\cdot v'=\mathrm{pack}(w)\\ \mathrm{Alph}(u')\cap\mathrm{Alph}(v')=\emptyset}} \mathbf{M}^*_{\mathrm{pack}(u')} \otimes \mathbf{M}^*_{\mathrm{pack}(v')}\\ =&\sum_{i\in I(\mathcal{A})} \Psi(\mathbf{F}_{f_1\cdots f_i}) \otimes \Psi(\mathbf{F}_{f_{i+1}\cdots f_n})\\ =&\Psi\otimes\Psi\circ\Delta(\mathbf{F}_f).\end{aligned}$$

Finally, let us prove that Ψ is surjective. Let w be a packed word. We denote by $\mathcal{P}(w)$ the set partition of the set $\{1, \cdots, n\}$

$$\mathcal{P}(w) = \bigcup_{0\le i\le \max(w)} \Big\{\{j; w_j = i\}\Big\}.$$

Then, the image, for the uniform block permutation $f = \mathrm{std}(w) \circ Id_{\mathcal{P}(w)}$, of $\mathbf{F}_f$ is $\mathbf{M}^*_w$. □

Example 5.13.

$$\begin{aligned}\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\12\end{smallmatrix}\right)} \cdot \mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}\Big) =&\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12 & 3\\12 & 3\end{smallmatrix}\right)} + \mathbf{F}_{\left(\begin{smallmatrix}13 & 2\\12 & 3\end{smallmatrix}\right)} + \mathbf{F}_{\left(\begin{smallmatrix}23 & 1\\12 & 3\end{smallmatrix}\right)}\Big)\\ =&\mathbf{M}^*_{112} + \mathbf{M}^*_{121} + \mathbf{M}^*_{211}\\ =&\mathbf{M}^*_{11} \cdot \mathbf{M}^*_1\\ =&\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}12\\12\end{smallmatrix}\right)}\Big) \cdot \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}\Big).\end{aligned}$$

We set $f = \left(\begin{array}{c|c|c|c} 1 & 24 & 3 & 5\\ 5 & 12 & 3 & 4\end{array}\right)$.

$$\begin{aligned}\Psi\otimes\Psi\circ\Delta(\mathbf{F}_f) =&1\otimes\Psi(\mathbf{F}_f) + \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}\Big) \otimes \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}13 & 2 & 4\\12 & 3 & 4\end{smallmatrix}\right)}\Big)\\ &+\Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}1 & 24 & 3\\4 & 12 & 3\end{smallmatrix}\right)}\Big) \otimes \Psi\Big(\mathbf{F}_{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}\Big) + \Psi(\mathbf{F}_f)\otimes 1\\ =&1\otimes\mathbf{M}^*_{41213} + \mathbf{M}^*_1\otimes\mathbf{M}^*_{1213} + \mathbf{M}^*_{3121}\otimes\mathbf{M}^*_1 + \mathbf{M}^*_{41213}\otimes 1\\ =&\Delta'(\mathbf{M}^*_{41213})\\ =&\Delta'(\Psi(\mathbf{F}_f)).\end{aligned}$$

5.2. **WQSym.** The Hopf algebra **WQSym** is the $\mathbb{K}$-vector space spanned by packed words endowed with the convolution product and the coproduct given by

$$\mathbf{M}_u \cdot \mathbf{M}_v = \sum_{\substack{w=u'\cdot v'\\ \mathrm{pack}(u')=u\\ \mathrm{pack}(v')=v}} \mathbf{M}_w, \tag{5.10}$$

$$\Delta(\mathbf{M}_u) = \sum_{0\le k\le\max(u)} \mathbf{M}_{u_{|\{1,\cdots,k\}}} \otimes \mathbf{M}_{\mathrm{pack}(u_{|\{k+1,\cdots,\max(u)\}})} \tag{5.11}$$

The dual $\mathbf{WQSym}'$ of $\mathbf{WQSym}^{*\prime}$ is the Hopf algebra endowed with the product

$$\mathbf{M}_u \cdot{}' \mathbf{M}_v = \sum_{\substack{w=u'\cdot v' \\ \mathrm{pack}(u')=u \\ \mathrm{pack}(v')=v \\ \mathrm{Alph}(u')\cap\mathrm{Alph}(v')=\emptyset}} \mathbf{M}_w \tag{5.12}$$

for any packed words u and v, and with the coproduct Δ. Indeed, if we denote by $w \in \mathbf{WQSym}^*$ the dual basis to $w \in \mathbf{WQSym}$, since the deconcatenation coproduct Δ' is multiplicity free, we have, by duality

$$\mathbf{M}_u \cdot{}' \mathbf{M}_v = \sum_{w : u\otimes v \in \Delta'(w)} \mathbf{M}_w. \tag{5.13}$$

This is equivalent to Equation (5.12). The Hopf algebra $\mathbf{WQSym}^{*\prime}$ and $\mathbf{WQSym}^*$ are isomorphic, Proposition 5.10, so is for $\mathbf{WQSym}'$ and $\mathbf{WQSym}$. We can explicitly give this isomorphism. The map Γ induces a dual map. This is

$$\Gamma^* : \begin{array}{ccc} \mathbf{WQSym}' & \longrightarrow & \mathbf{WQSym} \\ \mathbf{M}_w & \longmapsto & \displaystyle\sum_{u \preceq w} \mathbf{M}_u. \end{array}$$

Indeed, if we denote by $\mathbf{M}_w^* \in \mathbf{WQSym}^*$ the dual basis to $\mathbf{M}_w \in \mathbf{WQSym}$, since $\Gamma(\mathbf{M}_w^*)$ is a linear form, $\Gamma(\mathbf{M}_w^*)(u) = 1$ if $u \succeq w$, 0 otherwise, so is for $\mathbf{M}_w \circ \Gamma^*$. For example,

$$\Gamma^*(\mathbf{M}_{1212334}) = \mathbf{M}_{1212334} + \mathbf{M}_{1212223} + \mathbf{M}_{1212333} + \mathbf{M}_{1212222}.$$

Proposition 5.14. *The map*

$$\Phi : \begin{array}{ccc} \mathbf{WQSym}' & \longrightarrow & \mathbf{UBP} \\ \mathbf{M}_w & \longmapsto & \displaystyle\sum_{f : \Psi(f)=w} \mathbf{G}_f \end{array}$$

is an injective morphism of Hopf algebras.

Proof. The map Φ is the dual to Ψ. Since $\Psi(\mathbf{F}_f)(\mathbf{M}_w) = 1$ if $\Psi(f) = \mathbf{M}_w$, $\Psi(\mathbf{F}_f)(\mathbf{M}_w) = 0$ otherwise. Hence, $\mathbf{F}_f \circ \Phi(\mathbf{M}_w) = 1$ if $\Psi(f) = \mathbf{M}_w$ and $\mathbf{F}_f \circ \Phi(\mathbf{M}_w) = 0$ otherwise. $\square$

Example 5.15.

$$\begin{aligned}
\Phi(\mathbf{M}_{121} \cdot{}' \mathbf{M}_1) =& \Phi(\mathbf{M}_{1213} + \mathbf{M}_{1312} + \mathbf{M}_{2321}) \\
=& \Big(\mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 12 & 3 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 13 & 2 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 14 & 2 & 3 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 12 & 4 & 3 \end{array}\right)} \Big) \\
& + \Big(\mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 13 & 4 & 2 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 14 & 3 & 2 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 23 & 4 & 1 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 13 & 2 & 4 \\ 24 & 3 & 1 \end{array}\right)} \Big) \\
=& \Big(\mathbf{G}_{\left(\begin{array}{c|c} 13 & 2 \\ 12 & 3 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c} 13 & 2 \\ 13 & 2 \end{array}\right)} \Big) \cdot \mathbf{G}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} \\
=& \Phi(\mathbf{M}_{121}) \cdot \Phi(\mathbf{M}_1).
\end{aligned}$$

$$
\begin{aligned}
\Phi \otimes \Phi(\Delta(\mathbf{M}_{2131})) =& \Phi \otimes \Phi(1 \otimes \mathbf{M}_{2131} + \mathbf{M}_{11} \otimes \mathbf{M}_{12} + \mathbf{M}_{211} \otimes \mathbf{M}_{1} + \mathbf{M}_{2131} \otimes 1) \\
=& 1 \otimes \Big(\mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 2 & 13 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 3 & 12 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 2 & 14 & 3 \end{array}\right)} \Big) \\
& + \mathbf{G}_{\left(\begin{array}{c} 12 \\ 12 \end{array}\right)} \otimes \mathbf{G}_{\left(\begin{array}{c|c} 1 & 2 \\ 1 & 2 \end{array}\right)} + \Big(\mathbf{G}_{\left(\begin{array}{c|c} 1 & 23 \\ 2 & 13 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c} 1 & 23 \\ 3 & 12 \end{array}\right)} \Big) \otimes \mathbf{G}_{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)} \\
& + \Big(\mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 2 & 13 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 3 & 12 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 2 & 14 & 3 \end{array}\right)} \Big) \otimes 1 \\
=& \Delta \Big(\mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 2 & 13 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 3 & 12 & 4 \end{array}\right)} + \mathbf{G}_{\left(\begin{array}{c|c|c} 1 & 24 & 3 \\ 2 & 14 & 3 \end{array}\right)} \Big) \\
=& \Delta(\Phi(\mathbf{M}_{2131})).
\end{aligned}
$$

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